Force Flux and the Peridynamic Stress Tensor

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Abstract

The peridynamic model is a framework for continuum mechanics based on the idea that pairs of particles exert forces on each other across a finite distance. The equation of motion in the peridynamic model is an integro-differential equation. In this paper, a notion of a peridynamic stress tensor derived from nonlocal interactions is defined. At any point in the body, this stress tensor is obtained from the forces within peridynamic bonds that geometrically go through the point. The peridynamic equation of motion can be expressed in terms of this stress tensor, and the result is formally identical to the Cauchy equation of motion in the classical model, even though the classical model is a local theory. We also establish that this stress tensor field is unique in a certain function space compatible with finite element approximations.

Key words: Peridynamics, Dynamics, Elastic material, Stress, Flux, Equation of motion

1 Introduction

The peridynamic model [5] is an alternative theory of continuum mechanics based on integral, rather than differential, equations. The purpose of peridynamics is to provide a more general framework than the classical theory

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for problems involving discontinuities or other singularities in the deformation. The integral equations express a nonlocal force model that describes long-range material interaction. In this context, nonlocal means that particles separated by a finite distance may exert nonzero forces upon each other. This nonlocality is in contrast to the local force model intrinsic with classical continuum mechanics.

In the peridynamic model, the ideas of "force per unit area" and a stress tensor are not used. The goal of our paper is to define the *force flux* and the *peridynamic stress tensor* so establishing a closer connection between this and the classical view of continuum mechanics. We demonstrate that the peridynamic equation of motion

$$\rho(\mathbf{x})\ddot{\mathbf{u}}(\mathbf{x},t) = \int_{\mathcal{R}} \hat{\mathbf{f}}(\mathbf{u}(\mathbf{x}',t) - \mathbf{u}(\mathbf{x},t), \mathbf{x}' - \mathbf{x}, \mathbf{x})\, dV_{\mathbf{x}'} + \mathbf{b}(\mathbf{x},t) \tag{1}$$

when expressed in terms of the peridynamic stress tensor, is formally identical to the classical equation of motion, which is a partial differential equation. Our paper shows that the peridynamic stress tensor implicitly defines a formal Green's function for the differential equation

$$\nabla \cdot \boldsymbol{\nu}(\mathbf{x}) = \int_{\mathcal{R}} \hat{\mathbf{f}}(\mathbf{u}(\mathbf{x}',t) - \mathbf{u}(\mathbf{x},t), \mathbf{x}' - \mathbf{x}, \mathbf{x})\, dV_{\mathbf{x}'}. \tag{2}$$

Moreover, we show that a unique stress tensor field exists, satisfying an energy principle, within a certain function space compatible with finite element approximations.

The basic relation in the peridynamic model is the equation of motion (1) where $\mathbf{x}$ is a point in the reference configuration of a region $\mathcal{R}$, $\mathbf{u}$ is the displacement field, $\mathbf{b}$ is a prescribed body force density field, ρ is the reference density field, and $t \geq 0$ is the time. The vector-valued function $\hat{\mathbf{f}}$ is called the *pairwise force function*, whose value is the force density (with dimensions force/volume^2) that any point $\mathbf{x}'$ exerts on $\mathbf{x}$. The pairwise force function depends upon

$$\boldsymbol{\eta} = \mathbf{u}(\mathbf{x}',t) - \mathbf{u}(\mathbf{x},t), \quad \boldsymbol{\xi} = \mathbf{x}' - \mathbf{x},$$

the relative displacement and position vector between $\mathbf{x}$ and $\mathbf{x}'$, respectively, as well as $\mathbf{x}$ if the body is nonhomogeneous. Balance of linear and angular momenta places the following requirements on $\hat{\mathbf{f}}$:

$$\hat{\mathbf{f}}(-\boldsymbol{\eta}, -\boldsymbol{\xi}, \mathbf{x}+\boldsymbol{\xi}) = -\hat{\mathbf{f}}(\boldsymbol{\eta}, \boldsymbol{\xi}, \mathbf{x}), \qquad (\boldsymbol{\xi}+\boldsymbol{\eta}) \times \hat{\mathbf{f}}(\boldsymbol{\eta}, \boldsymbol{\xi}, \mathbf{x}) = \mathbf{0} \tag{3}$$

for all $\boldsymbol{\eta}$, all $\boldsymbol{\xi}$, and all $\mathbf{x} \in \mathcal{R}$. The function $\hat{\mathbf{f}}$ contains all constitutive information about the material. It is often convenient, although not an essential feature of the theory, to assume that if $\mathbf{x}$ and $\mathbf{x}'$ are separated in the reference configuration by a distance greater than some number $\delta > 0$ then the particles do not interact:

$$|\boldsymbol{\xi}| > \delta \quad \Longrightarrow \quad \hat{\mathbf{f}}(\boldsymbol{\eta}, \boldsymbol{\xi}, \mathbf{x}) = \mathbf{0}. \tag{4}$$

The number δ, if it exists for a particular material, is called the *horizon*.

2 Peridynamic Stress Tensor

Definition 1 *Let a peridynamic region $\mathcal{R}$ be given with pairwise force function $\hat{\mathbf{f}}$, and let $\mathbf{u}$ be the displacement field on $\mathcal{R}$. For a given $t \geq 0$, define a vector valued function $\mathbf{f} : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ by*

$$\mathbf{f}(\mathbf{p},\mathbf{q}) = \begin{cases} \hat{\mathbf{f}}(\mathbf{u}(\mathbf{p},t) - \mathbf{u}(\mathbf{q},t), \mathbf{p}-\mathbf{q}, \mathbf{q}) & \textit{if } \mathbf{p},\mathbf{q} \in \mathcal{R} \\ \mathbf{0} & \textit{otherwise.} \end{cases}$$

Thus $\mathbf{f}$ is the force density per unit volume squared that $\mathbf{p}$ exerts on $\mathbf{q}$, and $\mathbf{f}$ is called the *pairwise force density*. We remark that the constitutive model is supplied by $\hat{\mathbf{f}}$, in contrast to $\mathbf{f}$.

Define a set $\mathcal{I}$ consisting of ordered pairs of vectors in which the vectors equal each other:

$$\mathcal{I} = \left\{ (\mathbf{p},\mathbf{q}) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \mathbf{p} = \mathbf{q} \right\}. \tag{5}$$

The first of (3) and (1) imply that

$$\mathbf{f}(\mathbf{q},\mathbf{p}) = -\mathbf{f}(\mathbf{p},\mathbf{q}) \qquad \forall\, \mathbf{q},\mathbf{p} \in \mathbb{R}^3. \tag{6}$$

This further implies that $\mathbf{f} = \mathbf{0}$ on $\mathcal{I}$.

We assume, throughout this section, that $\mathbf{f}(\mathbf{x}',\mathbf{x})$ is Riemann-integrable. This assumption does *not* imply that $\mathbf{f}(\mathbf{p},\mathbf{q})$ is bounded as $|\mathbf{p}-\mathbf{q}| \to 0$. The example in the last section of this paper illustrates a material in which $\mathbf{f}$ is unbounded in this sense.

In the remainder of this paper, $\mathcal{S}$ denotes the unit sphere, and $d\Omega_{\mathbf{m}}$ denotes a differential solid angle on $\mathcal{S}$ in the direction of any unit vector $\mathbf{m}$.

Definition 2 *Let a deformation with displacement field $\mathbf{u}$ on a region $\mathcal{R}$ be given, and let $\mathbf{f}$ be the corresponding pairwise force density. Define the peridynamic stress tensor at any $\mathbf{x} \in \mathbb{R}^3$ by*

$$\boldsymbol{\nu}(\mathbf{x}) = \frac{1}{2} \int_{\mathcal{S}} \int_0^\infty \int_0^\infty (y+z)^2 \mathbf{f}(\mathbf{x}+y\mathbf{m}, \mathbf{x}-z\mathbf{m}) \otimes \mathbf{m}\, dz\, dy\, d\Omega_{\mathbf{m}}. \tag{7}$$

Definition 1 implies that

$$\mathbf{f}(\mathbf{x}+y\mathbf{m}, \mathbf{x}-z\mathbf{m}) = \hat{\mathbf{f}}(\mathbf{u}(\mathbf{x}+y\mathbf{m}) - \mathbf{u}(\mathbf{x}-z\mathbf{m}), (y+z)\mathbf{m}, \mathbf{x}-z\mathbf{m})$$

where we suppressed t for brevity.

The following result demonstrates a relationship between the peridynamic stress tensor and the pairwise force density.

Theorem 3 *Let a deformation with displacement field* $\mathbf{u}$ *on a region* $\mathcal{R}$ *be given, let* $\mathbf{f}$ *be the corresponding pairwise force density, and let* $\boldsymbol{\nu}$ *be given by Definition 2. If* $\mathbf{f}$ *is continuously differentiable on* $\mathbb{R}^3 \times \mathbb{R}^3 - \mathcal{I}$ *and if*

$$\mathbf{f}(\mathbf{p},\mathbf{q}) = o(|\mathbf{p}-\mathbf{q}|^{-2}) \quad \text{as} \quad |\mathbf{p}-\mathbf{q}| \to \infty, \tag{8}$$

then

$$\nabla \cdot \boldsymbol{\nu}(\mathbf{x}) = \int_{\mathcal{R}} \mathbf{f}(\mathbf{x}',\mathbf{x})\, dV_{\mathbf{x}'} \qquad \forall \mathbf{x} \in \mathbb{R}^3. \tag{9}$$

PROOF.

To make the notation more concise, define a vector-valued function $\mathbf{g}$ by

$$\mathbf{g}(\mathbf{m},y,z) = \mathbf{f}(\mathbf{x}+y\mathbf{m}, \mathbf{x}-z\mathbf{m}) \tag{10}$$

so that Definition 2 may be rewritten in terms of components in an orthonormal coordinate system as

$$\nu_{ij}(\mathbf{x}) = \frac{1}{2}\int_{\mathcal{S}}\int_0^\infty\int_0^\infty (y+z)^2 g_i(\mathbf{m},y,z) m_j \, dz\, dy\, d\Omega_{\mathbf{m}}. \tag{11}$$

Note for later use that (6) implies that

$$\mathbf{g}(-\mathbf{m},z,y) = -\mathbf{g}(\mathbf{m},y,z). \tag{12}$$

Observe from (10) and the chain rule that

$$\frac{\partial g_i}{\partial y} = m_j \frac{\partial f_i}{\partial p_j}, \qquad \frac{\partial g_i}{\partial z} = -m_j \frac{\partial f_i}{\partial q_j}, \qquad \frac{\partial g_i}{\partial x_j} = \frac{\partial f_i}{\partial p_j} + \frac{\partial f_i}{\partial q_j}$$

where the p_j and q_j refer to the first and second arguments of $\mathbf{f}$ as indicated in (1). Therefore,

$$m_j \frac{\partial g_i}{\partial x_j} = \frac{\partial g_i}{\partial y} - \frac{\partial g_i}{\partial z}. \tag{13}$$

By directly differentiating (11) and using (13),

$$\frac{\partial \nu_{ij}}{\partial x_j} = \frac{1}{2}\int_{\mathcal{S}}\int_0^\infty\int_0^\infty (y+z)^2 \left(\frac{\partial g_i}{\partial y} - \frac{\partial g_i}{\partial z}\right) dz\, dy\, d\Omega_{\mathbf{m}}. \tag{14}$$

Integration by parts leads to

$$
\begin{aligned}
\int_0^\infty (y+z)^2 \frac{\partial g_i}{\partial y}\, dy &= \int_0^\infty \frac{\partial}{\partial y}\left((y+z)^2 g_i\right)\, dy - 2\int_0^\infty (y+z) g_i\, dy \\
&= -z^2 g_i(\mathbf{m}, 0, z) - 2\int_0^\infty (y+z) g_i\, dy \qquad (15)
\end{aligned}
$$

where (8) has been used to drop the term arising from the upper limit of integration, *i.e.*, $g_i(\mathbf{m}, \infty, z) = 0$. Similarly,

$$
\int_0^\infty (y+z)^2 \frac{\partial g_i}{\partial z}\, dz = -y^2 g_i(\mathbf{m}, y, 0) - 2\int_0^\infty (y+z) g_i\, dz \qquad (16)
$$

Combining (14), (15), and (16), yields

$$
\begin{aligned}
\frac{\partial \nu_{ij}}{\partial x_j} &= \frac{1}{2}\int_{\mathcal{S}} \left(-\int_0^\infty z^2 g_i(\mathbf{m}, 0, z)\, dz + \int_0^\infty y^2 g_i(\mathbf{m}, y, 0)\, dy\right)\, d\Omega_{\mathbf{m}} \\
&= \frac{1}{2}\int_{\mathcal{S}} \left(\int_0^\infty z^2 g_i(-\mathbf{m}, z, 0)\, dz + \int_0^\infty y^2 g_i(\mathbf{m}, y, 0)\, dy\right)\, d\Omega_{\mathbf{m}} \\
&= \int_{\mathcal{S}} \int_0^\infty y^2 g_i(\mathbf{m}, y, 0)\, dy\, d\Omega_{\mathbf{m}} \qquad (17)
\end{aligned}
$$

where we have used the changes of variables $z \leftrightarrow y$, $\mathbf{m} \leftrightarrow -\mathbf{m}$, and (12). Recognizing (17) as a volume integral, and replacing g_i with f_i, we have that

$$
\frac{\partial \nu_{ij}}{\partial x_j} = \int_{\mathcal{S}} \int_0^\infty f_i(\mathbf{x} + y\mathbf{m}, \mathbf{x})(y^2\, dy\, d\Omega_{\mathbf{m}}) = \int_{\mathcal{R}} f_i(\mathbf{x}', \mathbf{x})\, dV_{\mathbf{x}'},
$$

and our result is established. $\square$

Remark 4 *The condition (8) on the decay of* $\mathbf{f}$ *is automatically satisfied by any material with a finite horizon.*

Remark 5 *The hypothesis and proof of Theorem 3 do not restrict the particular constitutive model that gives rise to the interparticle forces. In fact, it is not even necessary to assume that the material has a pairwise force function. For example, the force between any* $\mathbf{p}$ *and* $\mathbf{q}$ *could be influenced by multibody interactions. (In this case, Definition 1 would have to be modified.)*

Theorem 3 allows us to rewrite the peridynamic equation of motion (1) as

$$
\rho(\mathbf{x})\ddot{\mathbf{u}}(\mathbf{x}, t) = \nabla \cdot \boldsymbol{\nu}(\mathbf{x}, t) + \mathbf{b}(\mathbf{x}, t),
$$

which is formally identical to the equation of motion in the classical theory. The stress tensor $\boldsymbol{\nu}$ is the analogue of the Piola stress tensor in the classical theory.

To investigate the conditions under which $\boldsymbol{\nu}$ is symmetric, recall from the requirement of balance of angular momentum (second of (3)) that $\mathbf{f}$ is always parallel to the *deformed* bond direction $\hat{\mathbf{m}} = (\boldsymbol{\xi} + \boldsymbol{\eta})/|\boldsymbol{\xi} + \boldsymbol{\eta}|$. Therefore, the integrand in Definition 2 is symmetric when $\hat{\mathbf{m}} = \mathbf{m}$, which occurs if $\mathbf{u} \equiv \mathbf{0}$. So, $\boldsymbol{\nu}$ is symmetric if the deformed configuration and the reference configuration are the same. Otherwise, $\boldsymbol{\nu}$ is in general nonsymmetric (this is also true of the classical Piola stress tensor). In the classical model, the Piola stress tensor $\mathbf{S}$ can be transformed to a Cauchy stress tensor $\mathbf{T}$ through the relation $\mathbf{T} = \mathbf{S}\mathbf{F}^T/(\det\ \mathbf{F})$, where $\mathbf{F}$ is the deformation gradient tensor. However, in the peridynamic model, it is not assumed that $\mathbf{u}$ is continuously differentiable, so we cannot in general define a deformation gradient tensor. Therefore, although $\boldsymbol{\nu}$ is analogous to the Piola stress tensor $\mathbf{S}$, it is not possible in general to transform $\boldsymbol{\nu}$ into a Cauchy stress tensor.

3 Behavior of the Peridynamic Stress Tensor on a Boundary

Many deformations of practical interest involve $\mathbf{f}$ that fails to be continuously differentiable on $\partial\mathcal{R}$ as required by Theorem 3. The following demonstrates that the basic conclusion of Theorem 3 continues to hold even in this case.

Theorem 6 *Suppose that all conditions of Theorem 3 are met except that* $\mathbf{f}$ *is required to satisfy only the weaker condition that it be continuously differentiable on* $(\mathbb{R}^3 - \partial\mathcal{R}) \times (\mathbb{R}^3 - \partial\mathcal{R}) - \mathcal{I}$ *rather than on* $\mathbb{R}^3 \times \mathbb{R}^3 - \mathcal{I}$. *Then*

$$\nabla \cdot \boldsymbol{\nu}(\mathbf{x}) = \int_{\mathcal{R}} \mathbf{f}(\mathbf{x}', \mathbf{x})\, dV_{\mathbf{x}'} \qquad \forall \mathbf{x} \in \mathbb{R}^3 - \partial\mathcal{R}. \tag{18}$$

PROOF. Assume, temporarily, that $\mathcal{R}$ is convex. Consider $\mathbf{x} \in \mathcal{R} - \partial\mathcal{R}$. For this $\mathbf{x}$, let $\ell(\mathbf{m})$ denote the distance from $\mathbf{x}$ to $\partial\mathcal{R}$ along the direction $\mathbf{m}$. The exterior of $\mathcal{R}$ contributes nothing to the integral in Definition 2, so the limits of integration may be changed as follows:

$$\boldsymbol{\nu}(\mathbf{x}) = \frac{1}{2} \int_{\mathcal{S}} \int_0^{\ell(\mathbf{m})} \int_0^{\ell(-\mathbf{m})} (y+z)^2 \mathbf{f}(\mathbf{x} + y\mathbf{m}, \mathbf{x} - z\mathbf{m}) \otimes \mathbf{m}\, dz\, dy\, d\Omega_{\mathbf{m}}.$$

Upon differentiating to obtain the divergence as in (14), the Leibniz rule causes new terms to appear due to the possibly finite limits of integration over y and z:

$$\begin{aligned}\frac{\partial \nu_{ij}}{\partial x_j} = & \frac{1}{2} \int_{\mathcal{S}} \left\{ \int_0^{\ell(\mathbf{m})} \int_0^{\ell(-\mathbf{m})} (y+z)^2 \left(\frac{\partial g_i}{\partial y} - \frac{\partial g_i}{\partial z} \right) dz\, dy \right. \\ & + \int_0^{\ell(\mathbf{m})} (y + \ell(-\mathbf{m}))^2 g_i(\mathbf{m}, y, \ell(-\mathbf{m}))\, dy \\ & \left. - \int_0^{\ell(-\mathbf{m})} (\ell(\mathbf{m}) + z)^2 g_i(\mathbf{m}, \ell(\mathbf{m}), z)\, dz \right\} d\Omega_{\mathbf{m}}. \end{aligned} \tag{19}$$

The integration by parts in (15) also involves new terms because of the new limits of integration, for example,

$$\begin{aligned}\int_0^{\ell(\mathbf{m})} (y+z)^2 \frac{\partial g_i}{\partial y}\, dy &= \int_0^{\ell(\mathbf{m})} \frac{\partial}{\partial y}\left((y+z)^2 g_i\right)\, dy - 2\int_0^{\ell(\mathbf{m})} (y+z) g_i\, dy \\ &= (\ell(\mathbf{m})+z)^2 g_i(\mathbf{m}, \ell(\mathbf{m}), z) - z^2 g_i(\mathbf{m}, 0, z) \\ &\quad -2\int_0^{\ell(\mathbf{m})} (y+z) g_i\, dy. \qquad (20)\end{aligned}$$

Combining (19), (20), and the analogue of (19) for the integral over $\partial g_i/\partial z$ shows that the new terms arising from the boundary cancel each other out So, the remainder of the proof is the same as for Theorem 3. The case of $\mathbf{x}$ in the exterior of $\mathcal{R}$ is handled similarly, establishing the result (18). Any finite number of discontinuities in $\mathbf{g}$ along a direction $\mathbf{m}$ can be treated in the same way as shown above by defining $\{\ell_1(\mathbf{m}), \ell_2(\mathbf{m}), \dots\}$ at the locations of the discontinuities. Therefore, the conclusion holds for the case of nonconvex $\mathcal{R}$ as well as convex. $\square$

Remark 7 *Under the conditions of Theorem 6,* $\boldsymbol{\nu}$ *may fail to be differentiable on* $\partial\mathcal{R}$*. However, if we restrict the domain of* $\boldsymbol{\nu}$ *to* $\mathcal{R}+\partial\mathcal{R}$*, then the result (18) holds on this closed set. This is a familiar situation in the classical theory of continuum mechanics, in which a stress tensor field may be differentiable on* $\mathcal{R}$*, yet fail to be differentiable on* $\mathbb{R}^3$*.*

Remark 8 *The peridynamic stress tensor* $\boldsymbol{\nu}$ *may be non-null in the exterior of nonconvex* $\mathcal{R}$*, but* $\nabla\cdot\boldsymbol{\nu} = \mathbf{0}$ *in this exterior according to (18).*

Remark 9 *The same explanation of Remark 5 can be used to show that Theorem 6 does not require* $\mathbf{f}$ *to be a pairwise force function.*

We now address the behavior of the peridynamic stress tensor near $\partial\mathcal{R}$ and the exterior of $\mathcal{R}$.

Definition 10 *Let* $\mathcal{B}$ *be a closed, bounded, region in* $\mathbb{R}^3$ *of non-zero volume, and let* $\bar{\mathcal{B}}$ *denotes the convex hull of* $\mathcal{B}$*.*

The following result provides a boundary condition for the differential equation (2).

Theorem 11 *Let* $\mathbf{f}$ *be the pairwise force density resulting from a given displacement field* $\mathbf{u}$ *on* $\mathcal{B}$*, and let* $\boldsymbol{\nu}$ *be given by Definition 2. If* $\mathbf{n}(\mathbf{x})$ *denotes the outward-directed unit normal to* $\partial\mathcal{B}$ *at any* $\mathbf{x}\in\partial\mathcal{B}$*, then*

$$\boldsymbol{\nu}(\mathbf{x})\mathbf{n}(\mathbf{x}) = \mathbf{0} \qquad \forall\mathbf{x}\in\partial\bar{\mathcal{B}}.$$

PROOF. Consider any $\mathbf{x} \in \partial\bar{\mathcal{B}}$. Use Definition 2 in component form to obtain

$$\nu_{ij}n_j = \frac{1}{2}\int_{\mathcal{S}}\int_0^\infty\int_0^\infty (y+z)^2 f_i(\mathbf{x}+y\mathbf{m}, \mathbf{x}-z\mathbf{m}) m_j n_j \, dz \, dy \, d\Omega_{\mathbf{m}}. \tag{21}$$

Since $\bar{\mathcal{B}}$ is convex, any line segment whose endpoints are both in $\bar{\mathcal{B}}$ is contained entirely in $\bar{\mathcal{B}}$. In the integrand in (21), suppose $\mathbf{m}\cdot\mathbf{n} > 0$. Then, because $\mathbf{n}$ is an outward-directed unit normal, for sufficiently small $\Delta y > 0$, the point $\mathbf{x}+\Delta y\mathbf{m}$ is in the exterior of $\bar{\mathcal{B}}$. Therefore, because of the convexity of $\bar{\mathcal{B}}$, the entire half-line $\{\mathbf{x}+y\mathbf{m} \mid y > 0\}$ is in the exterior of $\bar{\mathcal{B}}$; establishing that the integrand in (21) vanishes for $\mathbf{m}\cdot\mathbf{n} > 0$. Similarly, it vanishes for $\mathbf{m}\cdot\mathbf{n} < 0$. The only remaining case is $\mathbf{m}\cdot\mathbf{n} = 0$, but since $m_j n_j$ appears in the integrand, in this case the integrand also vanishes. Hence the integrand vanishes for all $\mathbf{m}$ and our result is established. □

Remark 12 *If $\mathcal{B}$ is not convex, then $\boldsymbol{\nu}$ can be non-null at points in $\bar{\mathcal{B}} - \mathcal{B}$, even though no material is present there. But $\boldsymbol{\nu}$ must vanish in the exterior of $\bar{\mathcal{B}}$.*

Remark 13 *Theorem 11 does not require that* $\mathbf{f}$ *be continuously differentiable.*

Theorem 11 implies that the peridynamic stress tensor $\boldsymbol{\nu}$ implicitly defines a formal Green's function for the boundary value problem

$$\begin{aligned} \nabla\cdot\boldsymbol{\nu}(\mathbf{x}) &= \int_{\bar{\mathcal{B}}} \mathbf{f}(\mathbf{x}',\mathbf{x})\, dV_{\mathbf{x}'} \qquad && \mathbf{x}\in\bar{\mathcal{B}} \\ \boldsymbol{\nu}(\mathbf{x})\mathbf{n}(\mathbf{x}) &= \mathbf{0} && \mathbf{x}\in\partial\bar{\mathcal{B}}. \end{aligned} \tag{22}$$

4 Variational Interpretation of the Peridynamic Stress Tensor

Sections 2 and 3 presented a classical interpretation of the peridynamic stress tensor. This section establishes existence and uniqueness results for $\boldsymbol{\nu}$ in a variational sense. Such an interpretation allows us to identify function spaces associated with $\boldsymbol{\nu}$ and $\mathbf{f}$ so providing more general conditions than possible with a classical interpretation.

Define

$$\mathbf{F}(\mathbf{x}) = \int_{\mathcal{B}} \mathbf{f}(\mathbf{x}',\mathbf{x})\, dV_{\mathbf{x}'} \qquad \forall\mathbf{x}\in\mathcal{B}.$$

Let an orthonormal coordinate system be given, and let $\boldsymbol{\nu}$ be given by Definition 2. Define three vector fields $\boldsymbol{\nu}_1, \boldsymbol{\nu}_2, \boldsymbol{\nu}_3$ through

$$(\nu_1)_1 = \nu_{11}, \qquad (\nu_1)_2 = \nu_{12}, \qquad \ldots, \tag{23}$$

thus the components of each $\boldsymbol{\nu}_i$ are $\nu_{i1}, \nu_{i2}, \nu_{i3}$.

The balance of linear momentum (first of (3)) implies that the mean value of $F_i = F_i(\mathbf{x})$ over $\mathcal{B}$ is zero, so that we may choose

$$F_i \in L^2_0(\mathcal{B}) \equiv \{\, \psi \,|\, \psi \in L^2(\mathcal{B})\,,\ \int_{\mathcal{B}} \psi = 0 \,\}$$

where $L^2(\mathcal{B})$ is the space of square-integrable functions defined on $\mathcal{B}$ with respect to Lebesgue integration. The notation $[L^2(\mathcal{B})]^3$, used below, denotes the space of vector functions defined on $\mathcal{B} \subset \mathbb{R}^3$.

We rewrite (9) as

$$\int_{\mathcal{B}} \psi(\nabla \cdot \boldsymbol{\nu}_i)dV = \int_{\mathcal{B}} \psi F_i dV, \qquad \psi \in L^2(\mathcal{B}).$$

Standard results [4, pp. 586–587] give that

$$\boldsymbol{\nu}_i \in H_0(\mathrm{div}, \mathcal{B}) \equiv \{\, \mathbf{w} \,|\, \mathbf{w} \in L^2(\mathcal{B})^3,\ \nabla \cdot \mathbf{w} \in L^2(\mathcal{B}),\ \mathbf{w} \cdot \mathbf{n} = 0 \text{ on } \partial\mathcal{B} \,\}. \tag{24}$$

In words, a weak solution of the equation

$$\nabla \cdot \boldsymbol{\nu}_i = F_i, \quad \text{with} \quad \boldsymbol{\nu}_i \cdot \mathbf{n} = 0 \text{ on } \partial\mathcal{B}$$

such that

$$\|\boldsymbol{\nu}_i\|^2_{H(\mathrm{div},\mathcal{B})} = \|\boldsymbol{\nu}_i\|^2_{[L^2(\mathcal{B})]^3} + \|\nabla \cdot \boldsymbol{\nu}_i\|^2_{L^2(\mathcal{B})} < \infty$$

exists. The solution is unique up to a solenoidal function in $H_0(\mathrm{div}, \mathcal{B})$. A unique solution may be specified by the energy principle

$$\inf \frac{1}{2} \int_{\mathcal{B}} |\hat{\mathbf{w}}|^2, \quad \text{subject to} \quad \hat{\mathbf{w}} \in H_0(\mathrm{div}, \mathcal{B}) \quad \text{and} \quad \nabla \cdot \hat{\mathbf{w}} = F_i. \tag{25}$$

The energy principle, in effect, selects the (weak) solenoidal function of minimum energy—a unique member of $H_0(\mathrm{div}, \mathcal{B})$. This minimization problem is solved by introducing a Lagrange multiplier λ. The optimality system for the associated Lagrangian is: Find $(\mathbf{w}, \lambda) \in H_0(\mathrm{div}, \mathcal{B}) \times L^2_0(\mathcal{B})$ such that

$$\begin{aligned} (\mathbf{w}, \mathbf{s})_0 + (\nabla \cdot \mathbf{s}, \lambda)_0 &= 0 \qquad \forall \mathbf{s} \in H_0(\mathrm{div}, \mathcal{B}) \\ (\nabla \cdot \mathbf{w}, \psi)_0 &= (F_i, \psi)_0 \qquad \forall \psi \in L^2_0(\mathcal{B}), \end{aligned} \tag{26}$$

where

$$(\varphi, \psi)_0 \equiv \int \varphi(\mathbf{x})\psi(\mathbf{x})d\mathbf{x}$$

for $\varphi, \psi \in L^2(\mathcal{B})$. The first equation of (26) gives that

$$(\mathbf{w}, \mathbf{s})_0 = -(\nabla \cdot \mathbf{s}, \lambda)_0 \tag{27}$$

so that λ has a weak derivative. Therefore, applying Green's theorem to the first equation of (26), results in

$$(\mathbf{w} - \nabla\lambda, \mathbf{s})_0 = 0 \quad \forall \mathbf{s} \in H_0(\mathrm{div}, \mathcal{B}). \tag{28}$$

Selecting $\mathbf{w} = \boldsymbol{\nu}_i$ and

$$\lambda = \sum_{j=1}^{3} \int_0^{x_j} \nu_{ij}(\mathbf{r}) d\mathbf{r}$$

implies that $\boldsymbol{\nu}_i = \nabla\lambda$. The second equation of (26) is satisfied because of (9) so leading to the following result.

Theorem 14 *If* $\mathbf{f}$ *satisfies the conditions of Theorem 6 and* $\boldsymbol{\nu}$ *is given by Definition 2, then* $\boldsymbol{\nu}_i$ *defined by (23) satisfies the energy principle (25).*

Remark 15 *The proof of Theorem 14 is a standard argument for the dual formulation of the homogeneous Neumann problem*

$$\begin{aligned} \triangle\lambda(\mathbf{x}) &= F_i(\mathbf{x}), \quad \mathbf{x} \in \mathcal{B} \\ \mathbf{n} \cdot \nabla\lambda(\mathbf{x}) &= 0, \quad \mathbf{x} \in \partial\mathcal{B}, \end{aligned}$$

for example, see [4, pp. 586–588] or [2, p. 43].

Remark 16 *Theorem 14 does not employ the hypothesis that* $\mathbf{f}(\cdot,\cdot)$ *is a pairwise force function. See Remark 5.*

The variational interpretation gives the existence and uniqueness of a peridynamic stress tensor $\boldsymbol{\nu}$ under substantially more general conditions than Theorem 6. The force function $\mathbf{f}$ is only required to be an element of $L^2_0(\mathcal{B})$ so that differentiability of $\mathbf{f}$ is not assumed. Moreover, in contrast to Theorem 11, the boundary condition $\boldsymbol{\nu}\mathbf{n} = 0$ (in a weak sense) holds on $\partial\mathcal{B}$ regardless of the convexity of $\mathcal{B}$.

The variational interpretation allows us to exploit a relationship with the finite element method. The finite element solution of (26) requires a pair of suitable elements for the stress and Lagrange multiplier. The well-known elements of Raviart and Thomas [3] result in a stable finite element method for (26). The basis functions for the stress only satisfy continuity of the normal components across elements, and for the Lagrange multiplier are discontinuous across elements. The reader is referred to [2,4] for more information associated with the stable numerical solution of (26).

The tensor $\boldsymbol{\nu}$ and its finite element approximant $\boldsymbol{\nu}^h$ are not symmetric, as explained after Theorem 3. This is in contrast to the classical Hellinger-Reissner mixed formulation of the elasticity equations. The Hellinger-Reissner formulation requires that the stress be an element of $H_0(\mathrm{div}, \mathcal{B}; \mathbb{S})$ and the displacement in $L^2(\mathcal{B})$. The former space is the space of square-integrable symmetric tensors. The recent paper [1] describes the first stable finite discretization of the Hellinger-Reissner mixed formulation in three dimensions. We remark that the common engineering practice assumes a local force model, e.g. Cauchy Stress hypothesis, and a constitutive relation connecting stresses to strains resulting in a displacement based finite element method. The resulting nodal basis functions are continuous across elements.

5 Peridynamic Force Flux

Definition 17 *The peridynamic force flux vector at any* $\mathbf{x}$ *in the direction of any unit vector* $\mathbf{n}$ *is given by*

$$\boldsymbol{\tau}(\mathbf{x}, \mathbf{n}) = \boldsymbol{\nu}(\mathbf{x})\mathbf{n}.$$

Let $\mathcal{P}$ be a closed, bounded subregion in the interior of $\mathcal{B}$, and assume without loss of generality that $\mathbf{b} \equiv \mathbf{0}$ on $\mathcal{B}$. Let $\mathbf{L}$ be the total force on $\mathcal{P}$. Integrating both sides of (1) over $\mathcal{P}$,

$$\int_{\mathcal{P}} \rho \ddot{\mathbf{u}}(\mathbf{x}, t)\, dV_{\mathbf{x}} = \int_{\mathcal{P}} \int_{\mathcal{B}} \mathbf{f}(\mathbf{x}', \mathbf{x})\, dV_{\mathbf{x}'}\, dV_{\mathbf{x}}.$$

Suppose $\mathbf{f}$ is such that the conditions of Theorem 3 are satisfied. Newton's second law applied to the total momentum change within $\mathcal{P}$ therefore implies, using (9) and the divergence theorem,

$$\mathbf{L} = \int_{\mathcal{P}} \int_{\mathcal{B}} \mathbf{f}(\mathbf{x}', \mathbf{x})\, dV_{\mathbf{x}'}\, dV_{\mathbf{x}} = \int_{\mathcal{P}} \nabla \cdot \boldsymbol{\nu}(\mathbf{x})\, dV_{\mathbf{x}} = \int_{\partial\mathcal{P}} \boldsymbol{\tau}(\mathbf{x}, \mathbf{n})\, dA_{\mathbf{x}} \qquad (29)$$

where $\mathbf{n}$ is the outward-directed unit normal vector at any point $\mathbf{x} \in \partial\mathcal{P}$. Equation (29) shows that the total force on $\mathcal{P}$ is the surface integral of $\boldsymbol{\tau}$. This shows that $\boldsymbol{\tau}(\mathbf{x}, \mathbf{n})$ is, in this sense, the force per unit area exerted on a surface with normal vector $\mathbf{n}$ at $\mathbf{x}$ due to peridynamic interactions.

6 Mechanical Interpretation of the Force Flux

Definitions 2 and 17 yield

$$\boldsymbol{\tau}(\mathbf{x}, \mathbf{n}) = \frac{1}{2} \int_{\mathcal{S}} \int_0^\infty \int_0^\infty (y+z)^2 \mathbf{f}(\mathbf{x} + y\mathbf{m}, \mathbf{x} - z\mathbf{m})\mathbf{m} \cdot \mathbf{n}\, dz\, dy\, d\Omega_{\mathbf{m}}. \qquad (30)$$

Let $\mathbf{y} = \mathbf{x} + y\mathbf{m}$ and $\mathbf{z} = \mathbf{x} - z\mathbf{m}$. Consider the differential area $dA_{\mathbf{y}}$ on a sphere centered at $\mathbf{z}$ containing $\mathbf{y}$ that subtends a differential solid angle $d\Omega$ (Figure 1). Thus

$$dA_{\mathbf{y}} = (y+z)^2\, d\Omega.$$

The analogous quantity on a sphere centered at $\mathbf{y}$ containing $\mathbf{z}$ is identical:

$$dA_{\mathbf{z}} = (y+z)^2\, d\Omega.$$

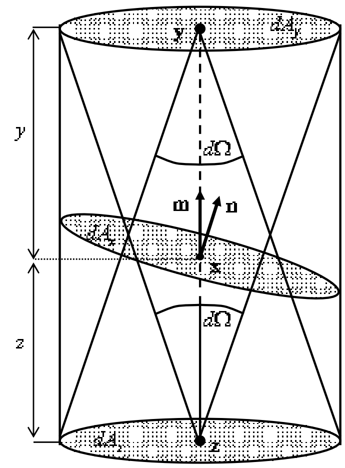

Fig. 1. Interpretation of the force flux at $\mathbf{x}$ across a plane with unit normal $\mathbf{n}$.

Let $dA_{\mathbf{x}}$ be the area on a plane with normal vector $\mathbf{n}$ through $\mathbf{x}$ that cuts through the cylinder of cross-sectional area $dA_{\mathbf{y}} = dA_{\mathbf{z}}$ with axis connecting $\mathbf{y}$ and $\mathbf{z}$:

$$dA_{\mathbf{x}} = \frac{(y+z)^2\ d\Omega}{\mathbf{m}\cdot\mathbf{n}}$$

where $\mathbf{m}$ is the unit vector pointing from $\mathbf{z}$ to $\mathbf{y}$. The total force that the volume element $dA_{\mathbf{y}}dy$ exerts on $dA_{\mathbf{z}}dz$ is

$$d\mathbf{L} = \mathbf{f}(\mathbf{y},\mathbf{z})((y+z)^2\ d\Omega\ dy)((y+z)^2\ d\Omega\ dz).$$

Thus, the differential force per unit area on the plane through $\mathbf{x}$ is

$$\frac{d\mathbf{L}}{dA_{\mathbf{x}}} = \frac{\mathbf{f}(\mathbf{y},\mathbf{z})((y+z)^2\ d\Omega)^2\ dy\ dz}{(y+z)^2\ d\Omega/\mathbf{m}\cdot\mathbf{n}} = \mathbf{f}(\mathbf{y},\mathbf{z})(y+z)^2(\mathbf{m}\cdot\mathbf{n})\ dy\ dz\ d\Omega. \quad (31)$$

Comparing this with the integrand in (30) leads to the physical interpretation of $\boldsymbol{\tau}$ as the force due to bonds that "go through" $\mathbf{x}$, per unit area of a plane with normal $\mathbf{n}$. The factor of 1/2 appears in (30) because the integral sums up both the forces on $\mathbf{z}$ due to $\mathbf{y}$ and those on $\mathbf{y}$ due to $\mathbf{z}$, which are of course equal in magnitude.

Our mechanical interpretation of the peridynamics stress is a close descendant of the definition of stress originally introduced in the early days of elasticity.

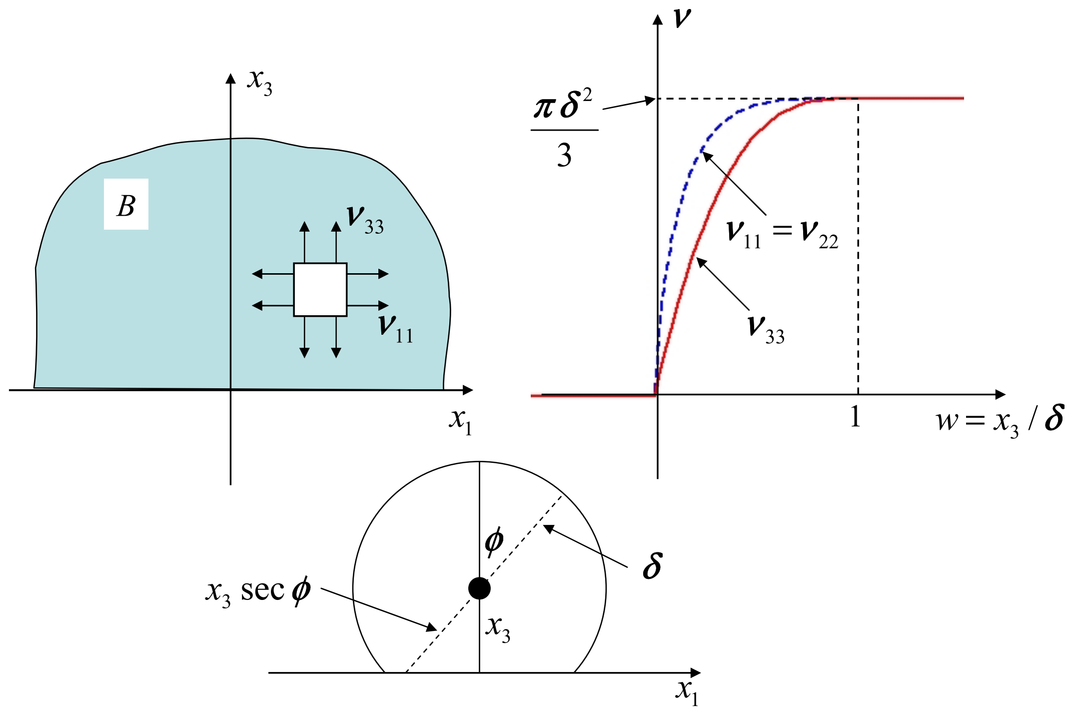

Fig. 2. Peridynamic stress components in a body occupying the upper half-space (Example 1).

According to Timoshenko [6], *The total stress on an infinitesimal element of a plane taken within a deformed elastic body is defined as the resultant of all the actions of the molecules situated on one side of the plane upon the molecules on the other, the directions of which (actions) intersect the element under consideration.* [2] Replacing *molecule* with peridynamic particle results in a definition that is consistent with our interpretation.

7 An example

Let $\mathcal{B}$ be a homogeneous body occupying the half-space $x_3 \geq 0$, and let $\delta > 0$. Let the pairwise force function by given by

$$\hat{\mathrm{f}}(\boldsymbol{\eta}, \boldsymbol{\xi}) = \frac{\boldsymbol{\xi} + \boldsymbol{\eta}}{|\boldsymbol{\xi} + \boldsymbol{\eta}|^3}, \qquad |\boldsymbol{\xi}| \leq \delta \tag{32}$$

and (4). Physically, this material is mechanically similar to a uniform distribution of gravitational mass, but with a cutoff distance δ for interactions. In

[2] See pages 108–109 of [6]. Timoshenko writes that this definition was due to Saint-Venant and was accepted by Cauchy.

the reference configuration, *i.e.*, $\mathbf{u} \equiv \mathbf{0}$, Definition 1 and (32) yield

$$\mathbf{f}(\mathbf{p}, \mathbf{q}) = \frac{\mathbf{p} - \mathbf{q}}{|\mathbf{p} - \mathbf{q}|^3}, \qquad |\mathbf{p} - \mathbf{q}| \leq \delta. \tag{33}$$

At points $\mathbf{x} \in \mathcal{B}$ sufficiently far the boundary ($x_3 > \delta$), applying (33) in Definition 2 leads to

$$\boldsymbol{\nu}(\mathbf{x}) = \frac{1}{2} \int_{\mathcal{S}} \mathbf{m} \otimes \mathbf{m} \int_0^{\delta} \int_0^{\delta - y} dz\ dy\ d\Omega_{\mathbf{m}}. \tag{34}$$

Using the spherical polar angles $m_1 = \sin\phi\cos\theta$, $m_2 = \sin\phi\sin\theta$, $m_3 = \cos\phi$, (34) may be evaluated as

$$\nu_{ij} = \int_0^{2\pi} \int_0^{\pi/2} m_i m_j \int_0^{\delta} \int_0^{\delta - y} \sin\phi\ dz\ dy\ d\phi\ d\theta \tag{35}$$

(see Figure 2). For points near the boundary ($0 \leq x_3 < \delta$), the limits of integration (35) must be altered:

$$\nu_{ij} = \int_0^{2\pi} \int_0^{\pi/2} m_i m_j \int_0^{\min\{\delta, x_3 \sec\phi\}} \int_0^{\delta - y} \sin\phi\ dz\ dy\ d\phi\ d\theta$$

A straightforward calculation results in the peridynamic stress tensor field components

$$\nu_{11} = \nu_{22} = \frac{\pi\delta^2}{3} \begin{cases} 0 & \text{if } w < 0 \\ \left(-\frac{3w}{2} + 3w^2 - \frac{w^3}{2} - 3w\log w \right) & \text{if } 0 \leq w < 1 \\ 1 & \text{if } 1 \leq w \end{cases}$$

$$\nu_{33} = \frac{\pi\delta^2}{3} \begin{cases} 0 & \text{if } w < 0 \\ \left(1 - (1 - w)^3 \right) & \text{if } 0 \leq w < 1 \\ 1 & \text{if } 1 \leq w \end{cases} \tag{36}$$

$$\nu_{12} = \nu_{21} = \nu_{23} = \nu_{32} = \nu_{31} = \nu_{13} = 0,$$

where $w = x_3/\delta$. These components are graphed in Figure 2. We also have

$$\nabla \cdot \boldsymbol{\nu} = \begin{pmatrix} 0 \\ 0 \\ \pi(\delta - x_3)^2 \end{pmatrix} \quad 0 \leq x_3 < \delta$$

and zero elsewhere.

Our example illustrates some of the properties of $\boldsymbol{\nu}$ that have been derived in this paper. These include

(1) $\nabla \cdot \boldsymbol{\nu}$ is continuously differentiable on $\mathbb{R}^3 - \partial\mathcal{B}$ but not on $\mathbb{R}^3$ as discussed in Remark 7,
(2) $\boldsymbol{\nu}\mathbf{n} = \mathbf{0}$ on $\partial\mathcal{B}$ as shown in Theorem 11,
(3) that $\mathbf{f}(\mathbf{p}, \mathbf{q})$ need not be bounded as $|\mathbf{p} - \mathbf{q}| \to 0$.

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